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ON THE FINITE TORSION AND RADIAL HEATING OF THERMOELASTIC CYLINDERS

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Abstract—The non-linear response to finite torsion accompanied by arbitrary radial heating of a cylinder of incompressible thermoelastic material with temperature-independent heat flux response is shown to be characterized completely by constitutive data collected from a block of the same material in a state of simple shear with uniform heating normal to the plane of shear.

1. INTRODUCTION

Simple shear and pure torsion in the absence of body force are among the finite deformations possible in every isotropic, incompressible elastic material [2], and it may be shown rigorously [10] that the constitutive data collected from a block of elastic material subjected to simple shear are sufficient to describe completely the behavior of a cylinder of the same material in pure torsion. We consider in the present work to what extent these well-known observations may be extended to general thermoelasticity.

While a state of simple shear with a uniform temperature gradient normal to the plane of shear may be held with surface tractions and heat fluxes alone in every isotropic, incompressible thermoelastic material (of the restricted class delineated in Section 2 below), there is no single temperature field that may be so coupled with a state of pure torsion (see [7]). Nevertheless, we show (in Section 4) that stress and heat flux measurements made in the controllable state of simple shear with heating (described in Section 3) provide data sufficient to give an ordinary differential equation for the temperature field in a cylinder whose cylindrical surfaces are isothermal. It then follows that the thermoelastic problem of pure torsion with radial heating may be solved completely, whether or not a nonconductive heat supply is present. However, the constitutive data available from a simple shearing with heating experiment are found *not* to be sufficient to characterize the behavior of a cylinder subjected to pure torsion coupled with either an *axial* or *circumferential* temperature gradient.

We note that many of the non-classical properties of elastic response to finite simple shear [10] are qualitatively unchanged by the presence of a uniform temperature gradient normal to the plane of shear—but there are notable exceptions, which are observed in Section 3. For example, while it is always possible to free of normal stress *any* pair of opposite faces of a simply sheared block, this is not the case when heating accompanies the shear, and only the isothermal faces may be so freed. The thermoelastic response to pure torsion with radial heating is likewise found to be qualitatively the same as the purely mechanical counterpart, and this similarity is considered in Section 4.

The central observation, that *all* of the thermoelastic invariants associated with a state of pure torsion with radial heating occur in exactly the same relationships as those in simple shear with heating, is made in Section 4. It is this coincidence that enables one to predict the thermomechanical response of a twisted and heated cylinder with only the constitutive data

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collected from a sheared and heated block of the same material. Recently published catalogs of thermomechanical states possessing invariants related as are those of controllable states do not contain the torsion state, however, for these lists are restricted to states with constant strain invariants [5] or to states whose principal stretches coincide with coordinate lines [6], and torsion falls into neither of these catagories. Pure torsion coupled with a temperature field proportional to $\ln r$ has recently been shown [1] to be possible in all materials of a special class of those we consider, and Laws [3] has shown that a state of pure torsion may always coexist with a heat flux field which varies inversely as the radius in another restricted class of thermoelastic materials. However, since we do not wish to restrict our analysis *a priori* to materials more specific than those delineated in Section 2 below, and since we wish to control the temperature rather than the heat flux field, none of these previous investigations directly applies to the present work.

Our notation follows closely, but not completely, that of "The non-linear field theories" [10], where one can find the foundations of the theory of thermoelasticity that we assume. In general, the mathematical language is that of general tensor analysis, in which repeated indices, once co- and once contra-variant, are to be summed over 1, 2, 3.

2. EQUATIONS OF THERMOELASTICITY

Deformations of continuous media are given by $x^i = x^i(X^J)$, where x^i and X^J are curvilinear coordinates of the positions of a material point in the deformed and reference states, respectively. The left Cauchy–Green tensor is an appropriate finite strain measure, and components of it and its inverse are given by

$$B^{ij} = G^{JK}\frac{\partial x^i}{\partial X^J}\frac{\partial x^j}{\partial X^K}, \tag{2.1}$$

$$(\bar{B}^1)_{ij} = G_{JK}\frac{\partial X^J}{\partial x^i}\frac{\partial X^K}{\partial x^j}, \tag{2.2}$$

where G_{JK} are components of the metric of the X^J-coordinate system. A temperature field in a deformed medium is given by $\tau = \tau(x^i)$, and the following five scalar invariants are associated with a volume-preserving thermomechanical state (see [10, 11]):

$$\begin{aligned} I_1 &= B_i^i, \\ I_2 &= (\bar{B}^1)_i^i, \\ I_3 &= \tau_{,i}\tau^{,i}, \\ I_4 &= \tau_{,i}B_j^i\tau^{,j}, \\ I_5 &= \tau_{,i}(\bar{B}^1)_j^i\tau^{,j}. \end{aligned} \tag{2.3}$$

We should emphasize that in our notation the third invariant measures the magnitude of the temperature gradient, whereas in the general literature it more commonly measures volume change through the determinant of B_j^i. Since we shall be concerned exclusively with incompressible (see [4], Section 5.7) materials, for which $\det(B_j^i) \equiv 1$ for all admissible deformations, however, that third kinematical invariant plays no explicit role in our analysis.

Steady-state thermoelasticity is governed by four field equations expressing balance of momentum and energy, viz.

$$t^{ij}_{,j} + b^i = 0 \tag{2.4}$$

and

$$q^i_{,i} - h = 0, \tag{2.5}$$

where the body force b^i and the non-conductive heat supply h are measured per unit volume, and where the symmetric stress tensor t^{ij} and the heat conduction vector q^i are assumed given by the constitutive equations (see [10], Section 96)

$$t_j^i = -p\delta_j^i + \Psi_1 B_j^i + \Psi_{-1}(\bar{B}^1)_j^i \tag{2.6}$$

and

$$q^i = [\phi_1 B_j^i + \phi_0 \delta_j^i + \phi_{-1}(\bar{B}^1)_j^i]\tau^{,j}, \tag{2.7}$$

which characterize the class of homogeneous, isotropic, incompressible thermoelastic materials with which we are concerned. Here $p = p(x^i)$ is a hydrostatic pressure and δ_j^i is the Kronecker delta. The material response functions Ψ_Γ ($\Gamma = \pm 1$) are derivable from a material's free energy density function

$$\psi = \psi(\tau, I_1, I_2) \tag{2.8}$$

according to

$$\Psi_1 = 2\rho \frac{\partial \psi}{\partial I_1},$$

$$\Psi_{-1} = -2\rho \frac{\partial \psi}{\partial I_2},$$

where ρ is the (uniform) density, and hence

$$\Psi_\Gamma = \Psi_\Gamma(\tau, I_1, I_2). \tag{2.9}$$

The heat-flux-response coefficients ϕ_Λ ($\Lambda = -1, 0, 1$) depend on the invariants (2.3) as follows:

$$\phi_\Lambda = \phi_\Lambda(I_1, I_2, I_3, I_4, I_5). \tag{2.10}$$

If the ϕ_Λ depend explicitly also on the temperature τ, the results we are about to obtain do not necessarily follow (see [7]). We shall henceforth take (2.6) and (2.7) with (2.9) and (2.10) as defining the class of thermoelastic materials with which we shall be concerned.

3. SIMPLE SHEAR WITH NORMAL HEATING

It has been established [7] that the uniform temperature gradient and homogeneous deformation

$$\tau = \kappa z \quad \text{and} \begin{cases} x = \dfrac{C}{\sqrt{A}} X - \dfrac{D}{\sqrt{B}} Y, \\ y = \dfrac{D}{\sqrt{A}} X + \dfrac{C}{\sqrt{B}} Y, \\ z = \sqrt{AB} Z, \end{cases} \tag{3.1}$$

where the principal stretches $1/A > 0$ and $1/B > 0$ and the temperature gradient κ are arbitrary constants, and where the constants C and D satisfy $C^2 + D^2 = 1$, may be held by suitable surface tractions and heat fluxes alone in all materials of the kind defined by (2.6) and (2.7). In other words, the thermomechanical state (3.1) satisfies (2.1), with $b^i \equiv 0$ and t_j^i given by (2.6) and (2.2), with $h \equiv 0$ and q^i given by (2.7), identically in the material response functions Ψ_Γ and ϕ_Λ of (2.9) and (2.10).

Hence it follows *a fortiori* that it is always possible to maintain in such materials a state of simple shear of amount K with a uniform temperature gradient normal to the plane of shear,

$$\tau = \kappa z \quad \text{and} \quad \begin{cases} x = X + KY, \\ y = Y, \\ z = Z, \end{cases} \tag{3.2}$$

for this represents the special case of (3.1) corresponding to a reorientation of the coincident coordinate systems (x, y, z) and (X, Y, Z) and the choices

$$A = \frac{1}{B} = 1 + \frac{1}{2}K^2 \pm K\sqrt{1 + \frac{1}{4}K^2},$$

$$C^2 = 1 - D^2 = \left(1 + \frac{1}{4}K^2\right)^{-1}.$$

The kinematics of simple shear are treated exhaustively in ([11] Section 45), and an extensive treatment of the non-linear response of elastic materials to simple shear, after which we model the present work, appears in ([10] Sections 54, 56).

For simple shear we have, by (2.1) and (2.2),

$$\|B_j^i\| = \begin{Vmatrix} 1 + K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{3.3}$$

$$\|(\bar{B}^1)_j^i\| = \begin{Vmatrix} 1 & -K & 0 \\ -K & 1 + K^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

and, therefore, since

$$\tau_{,i} = (0, 0, \kappa),$$

the invariants (2.3) associated with (3.2) are

$$\begin{aligned} I_1 &= I_2 = 3 + K^2, \\ I_3 &= I_4 = I_5 = \kappa^2. \end{aligned} \tag{3.4}$$

The equilibrium equation (2.4) with $i = 3$ and $b^i \equiv 0$ is satisfied by having

$$p = p(z) = -p_0 + \tilde{\Psi}_1 + \tilde{\Psi}_{-1} \tag{3.5}$$

where p_0 is an arbitrary constant and where

$$\tilde{\Psi}_\Gamma = \tilde{\Psi}_\Gamma(\tau, K^2) \equiv \Psi_\Gamma(\tau, 3 + K^2, 3 + K^2) \tag{3.6}$$

depends on z. Then the non-zero physical components of the stress tensor, which satisfy (2.4) with $b^i \equiv 0$ identically, are

$$\begin{aligned} t\langle xx\rangle &= p_0 + K^2\tilde{\Psi}_1 \\ t\langle yy\rangle &= p_0 + K^2\tilde{\Psi}_{-1} \\ t\langle zz\rangle &= p_0 \\ t\langle xy\rangle &= K\tilde{\mu} \end{aligned} \tag{3.7}$$

where

$$\tilde{\mu} = \tilde{\mu}(\tau, K^2) \equiv \tilde{\Psi}_1(\tau, K^2) - \tilde{\Psi}_{-1}(\tau, K^2) \tag{3.8}$$

is an even function of K and depends on z through the temperature field τ. Following [10], we call $\tilde{\mu}$ the *generalized temperature-dependent shear modulus.* The *universal relation*

$$Kt\langle xy\rangle = t\langle xx\rangle - t\langle yy\rangle \tag{3.9}$$

holds and demonstrates that the Poynting effect of unequal normal stresses will generally be present in all but the degenerate materials for which $\tilde{\mu} = 0$ for arbitrary (τ, K^2). The shearing and normal tractions on the inclined faces $x - Ky = \text{const}$ of an initially rectangular sheared block are calculated in ([10] Section 54) to be given by, respectively,

$$T = t\langle xy\rangle/(1+K^2)$$

and

$$N = t\langle yy\rangle - Kt\langle xy\rangle/(1+K^2) \tag{3.10}$$

and these apply also to the thermoelastic block with the temperature-dependent stresses given by (3.7). As in the purely mechanical case, the Poynting effect $N \neq t\langle yy\rangle$ also appears on the bounding planes of a sheared and heated rectangular block.

However, unlike in the purely mechanical case, where any pair of parallel faces may be simultaneously freed of normal stress, we see that in the thermomechanical simple shearing (3.2), *only the isothermal faces $z = \text{const}$ of the sheared block may be simultaneously freed of traction.* This follows from (3.7) and (3.10) and the fact that the $\tilde{\Psi}_\Gamma$, and therefore $\tilde{\mu}$, depend on z through the temperature field τ, and it follows, furthermore, that the Poynting effects are complicated by the fact that the normal stresses are not uniform over non-isothermal surfaces.

The heat flux response in the state (3.2) is, by (2.7),

$$\begin{aligned} q^x &= q^y = 0, \\ q^z &= \tilde{\nu}(\kappa^2, K^2)\kappa, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} \tilde{\nu} &= \tilde{\nu}(\kappa^2, K^2) \\ &\equiv \tilde{\phi}_1(\kappa^2, K^2) + \tilde{\phi}_0(\kappa^2, K^2) + \tilde{\phi}_{-1}(\kappa^2, K^2), \end{aligned} \tag{3.12}$$

and where

$$\tilde{\phi}_\Lambda = \tilde{\phi}_\Lambda(\kappa^2, K^2) \equiv \phi_\Lambda(3+K^2, 3+K^2, \kappa^2, \kappa^2, \kappa^2) \tag{3.13}$$

for $\Lambda = -1, 0, 1$. We call $\tilde{\nu}$ the *generalized thermal conductivity in simple shear.* We note that there is no heat flow across the faces normal to the isothermal faces $z = \text{const}$ of the sheared block, and, as we would expect, the conductivity is an even function of the amount of shear K. Finally we observe that the fluxes (3.11) satisfy (2.5) with $h = 0$ identically in the material coefficients ϕ_Λ of (2.10).

Although, as we have seen, the tractions necessary to effect simple shearing with heating are not as simple as those required to produce a state of purely mechanical simple shear, the thermomechanical state (3.2) *is* controllable. Therefore, it is theoretically possible to subject *any* thermoelastic material characterized by the response functions Ψ_Γ and ϕ_Λ to such a state, to measure the shearing traction $t\langle xy\rangle$, one normal traction, $t\langle yy\rangle$ or N, say, and the heat flux q^z necessary to maintain various combinations of temperature gradient and amount of shear, and thereby to determine the generalized temperature-dependent shear modulus $\bar{\mu}$ and the generalized thermal conductivity in simple shear $\bar{\nu}$. We shall now show that these constitutive data alone are sufficient to solve problems involving the torsion and radial heating of cylinders.

4. PURE TORSION WITH RADIAL HEATING

The thermomechanical state of pure torsion (see [10] Section 57) accompanied by a radial temperature gradient is given in circular cylindrical coordinates by

$$\tau = \tau(r) \quad \text{and} \quad \begin{cases} r = R, \\ \theta = \Theta + DZ, \\ z = Z \end{cases} \tag{4.1}$$

where D measures the amount of twist and where we leave the axisymmetric, non-uniform temperature field indefinite for the present. By (2.1) and (2.2) we have

$$\|B_j^i\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1+D^2r^2 & D \\ 0 & Dr^2 & 1 \end{Vmatrix} \tag{4.2}$$

$$\|(\bar{B}^1)_j^i\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & -D \\ 0 & -Dr^2 & 1+D^2r^2 \end{Vmatrix},$$

and, since

$$\tau_{,i} = (\tau'(r), 0, 0),$$

where the prime denotes differentiation with respect to r, the associated invariants (2.3) are

$$\begin{aligned} I_1 &= I_2 = 3 + D^2r^2, \\ I_3 &= I_4 = I_5 = \kappa^2 \end{aligned} \tag{4.3}$$

where $\kappa = \kappa(r) = \tau'(r)$. That the strain invariants are equal, as in simple shearing, is well-known, but the fact that *all five thermomechanical invariants occur in the same relationships in pure torsion with radial heating as in simple shearing with normal heating* seems not to have been observed in the literature. From this observation it follows, as indicated at the end of Section 3, that *the constitutive data collected from a block of thermoelastic material subjected to simple shear with normal heating is sufficient to study the problem of pure torsion and radial heating of a cylinder of the same material.* However, unlike the purely mechanical state (or the isothermal thermomechanical state) of pure torsion, which is known to be possible in *all* perfectly elastic materials (see [2, 9]), there is no single non-uniform temperature field for which the thermomechanical state (4.1) is possible in all thermoelastic materials without the presence of a non-conductive heat supply [7]. It will follow from our work, however, that the heat supply necessary to hold a given temperature field in a particular material may be readily derived from only the constitutive data of a simple-shear-with-heating experiment (see equation (4.9) below).

Using (4.2) in (2.6) gives the components of the stress tensor, all of which depend on r. The first equilibrium equation of (2.4) with zero body force is satisfied by having

$$p = p(r) = -p_0 + \tilde{\Psi}_1 + \tilde{\Psi}_{-1} - D^2 \int r\tilde{\Psi}_1 \, dr \tag{4.4}$$

where p_0 is an arbitrary constant of integration, which may be chosen so as to free *one* cylindrical surface of traction, and where

$$\tilde{\Psi}_\Gamma = \tilde{\Psi}_\Gamma(\tau(r), D^2r^2) \tag{4.5}$$

are the same functions as in (3.6), and which, therefore, may be determined in a simple-shearing-with-normal-heating experiment. The remaining equilibrium equations (2.4) are satisfied identically in the Ψ_Γ by the following state of stress:

$$\begin{aligned} t\langle rr\rangle &= p_0 - D^2 \int r\tilde{\Psi}_1 \, dr \\ t\langle\theta\theta\rangle &= t\langle rr\rangle + D^2r^2\tilde{\Psi}_1 \\ t\langle zz\rangle &= t\langle rr\rangle + D^2r^2\tilde{\Psi}_{-1} \\ t\langle\theta z\rangle &= Dr\tilde{\mu} \\ t\langle r\theta\rangle &= t\langle zr\rangle = 0 \end{aligned} \tag{4.6}$$

where

$$\tilde{\mu} = \tilde{\mu}(r) = \tilde{\mu}(\tau(r), D^2r^2)$$

is the generalized temperature-dependent shear modulus of (3.8). The *universal relation*

$$Drt\langle\theta z\rangle = t\langle\theta\theta\rangle - t\langle zz\rangle$$

holds independent of the material response functions, and the presence, in general, of a non-vanishing axial stress $t\langle zz\rangle$ demonstrates the existence of a *Poynting effect for torsion.* The only difference between the thermomechanical stress state (4.6) and the purely mechanical state exhibited in ([10] equation 47.18) is that in (4.6) the response functions depend on the temperature as well as the strain invariants. Since these independent variables all depend on the same radial coordinate as a parameter, the resultant torque T and axial force N, which are calculated in [10] as being necessary to twist and maintain the length of the cylinder, apply *mutatis mutandis* here, and we have that

$$\begin{aligned} T &= 2\pi D \int_{r_1}^{r_2} r^3 \tilde{\mu}(\tau(r), D^2r^2) \, dr, \\ N &= -\pi\left[-r^2 t\langle rr\rangle\Big|_{r_1}^{r_2} + D^2 \int_{r_1}^{r_2} r^3(\tilde{\Psi}_1 - 2\tilde{\Psi}_{-1}) \, dr\right] \end{aligned} \tag{4.7}$$

where r_1 and r_2 are the inner and outer radii of the cylinder.

The heat flux field associated with (4.1) is readily calculated from (2.7) and (4.2) to be

$$\begin{aligned} q^r &= (\tilde{\phi}_1 + \tilde{\phi}_0 + \tilde{\phi}_{-1})\tau'(r) = \tilde{\nu}(r)\tau'(r), \\ q^\theta &= q^z = 0, \end{aligned} \tag{4.8}$$

where $\tilde{\phi}_\Lambda = \tilde{\phi}_\Lambda(r) = \tilde{\phi}_\Lambda(\kappa(r)^2, D^2r^2)$ and where

$$\tilde{\nu} = \tilde{\nu}(r) = \tilde{\nu}(\kappa(r)^2, D^2r^2)$$

are the same material functions defined in (3.12) and (3.13), which are completely determinable in the controllable state of simple shear with normal heating. If $h = h(r)$ is assumed, the remaining field equation (2.5) may be put in the form of an ordinary differential equation:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\tilde{\nu}(r)\frac{\mathrm{d}\tau(r)}{\mathrm{d}r}\right) = h(r), \tag{4.9}$$

but there is no *single* temperature field $\tau = \tau(r)$ that satisfies (4.9) for arbitrary $\tilde{\nu}$ and $h \equiv 0$ (see [7]). Although it has been observed recently [1] that, if the ϕ_Λ and, therefore, $\tilde{\nu}$ are constants, it is always possible when $h \equiv 0$ to couple pure torsion with the temperature field $\tau = k \ln r$, where k is a constant, since we should not like to narrow here the class of thermoelastic materials beyond the assumptions expressed in (2.7) and (2.10), we may assert that *the thermomechanical state of pure torsion and radial heating* (4.1) *is possible in an incompressible thermoelastic cylinder whose heat flux response is independent of temperature provided* $\tau = \tau(r)$ *solves the differential equation* (4.9), *where* $\tilde{\nu}(r)$ *may always be determined completely through the simple shear and heating of a block of the same material.* Hence we have the potential of solving a class of problems, with or without heat generation, in non-linear thermoelasticity.

As we have observed, the stress field depends only on the radial coordinate and, therefore, has all the properties associated with purely mechanical torsion (see [10] Section 57). The heat flux field, although qualitatively unaffected by the presence of a twist D, is quantitatively affected through the generalized thermal conductivity in shear $\tilde{\nu}$. We paraphrase the "capital result" of ([10] Section 57): *Measurement of the shear stress, any one non-uniform normal stress, and the heat flux (along with the resultant amount of shear and temperature gradient) in the simple shear and normal heating of an isotropic, incompressible thermoelastic material whose heat flux response is independent of temperature suffices to determine the behavior of a cylinder of the same material in pure torsion with radial heating.* If one chooses to specify *a priori* the temperature field $\tau = \tau(r)$ in (4.1), then (4.9) with the simple-shear constitutive data suffices to calculate the heat generation $h = h(r)$ necessary to maintain that thermomechanical state.

Finally, we note that if pure shear is coupled with $\tau = \tau(\theta)$ or $\tau = \tau(z)$, the invariants (2.3) do not have the same relationships as they do in simple shear with normal heating. Therefore, simple-shear-with-normal-heating experiments will *not* suffice to determine the response of cylinders to pure torsion with axial or circumferential temperature gradients. Nevertheless, if one assumes constitutive data are collected from more than just simple shearing experiments, these problems may be approached.

While the problem of torsion and extension coupled with axial heating of a cylinder has been considered by Pipkin and Rivlin [8] (see [10], p. 359), this state is not controllable in materials of the class we have considered, and the invariants do not occur in the relations possessed by those associated with the state of simple shear with normal heating. Parkus [4] has considered the problem of pure torsion coupled with axial heating, but only for materials for which the stress response is independent of temperature.

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